MJC-04

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Online class

## Homogeneous Differential Equations :

**Homogeneous function :** A function $F(x,y)$ is said to be homogeneous function of degree $n$ if
$F(\alpha x, \alpha y) = \alpha^n F(x,y)$ for any nonzero constant $\alpha$.

**Examples** $n \in \mathbb{N} \cup \{0\}$

① $F_1(x,y) = y^2 + 2xy$

$F_1(\alpha x, \alpha y) = (\alpha y)^2 + 2(\alpha x)(\alpha y)$

$\Rightarrow F_1(\alpha x, \alpha y) = \alpha^2 [y^2 + 2xy]$

$\Rightarrow F_1(\alpha x, \alpha y) = \alpha^2 F_1(x,y)$

Hence $F_1$ is a homogeneous function of degree 2.

② $F_2(x,y) = 2x - 3y$,

$F_2(\alpha x, \alpha y) = 2\alpha x - 3\alpha y$

$\Rightarrow F_2(\alpha x, \alpha y) = \alpha(2x - 3y)$

$\Rightarrow F_2(\alpha x, \alpha y) = \alpha^1 F_2(x,y)$

$\Rightarrow$ $F_2$ is a homogeneous function of degree 1.

③ $F_3(x,y) = \cos\left(\frac{y}{x}\right)$, $F_3(\alpha x, \alpha y) = \cos\left(\frac{\alpha y}{\alpha x}\right)$

$\Rightarrow F_3(\alpha x, \alpha y) = \cos\left(\frac{y}{x}\right)$

$\Rightarrow$ $F_3$ is a homogeneous function of degree 0.

H.W. Is $F(x,y) = \sin x + \cos y$ homogeneous function? Justify your answer.

- A differential equation of the form $\frac{dy}{dx} = F(x,y)$ is said to be homogeneous if $F(x,y)$ is a homogeneous function of degree zero.
- Solution of the homogeneous diff equation of the Type $\frac{dy}{dx} = F(x,y) = g\left(\frac{y}{x}\right)$ — (A)

Step 1. Substitute $y = vx$ — (1)

Step 2. Differentiate eqn (1) with respect to $x$, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad -(2)$$

Step 3. ~~Substi~~ From eqns (1) and (2), we get

$$v + x\frac{dv}{dx} = g\left(\frac{vx}{x}\right)$$

$$\Rightarrow v + x\frac{dv}{dx} = g(v)$$

$$\Rightarrow x\frac{dv}{dx} = g(v) - v$$

$$\Rightarrow \frac{dv}{g(v)-v} = \frac{dx}{x} \quad \text{[variable separable form]} \quad -(3)$$

Step 4. Integrate eqn (3) both sides, we get

$$\int \frac{dv}{g(v)-v} = \log x + C.$$

Example(1) show that the differential equation $(x-y)\frac{dy}{dx} = x+2y$ is homogeneous and Solve it.

Solution: Given equation can be written as

$$\frac{dy}{dx} = \frac{x+2y}{x-y} = F(x,y) \quad -(1)$$

Replace $x$ by $\lambda x$ and $y$ by $\lambda y$, then

$$F(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \frac{\lambda(x+2y)}{\lambda(x-y)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \lambda^0 \frac{x+2y}{x-y} = \lambda^0 F(x,y)$$

$\Rightarrow$ Diff eqn (1) is a homogeneous diff eqn.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Eqn (1) $\Rightarrow$ $v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \frac{(1-v)dv}{1+v+v^2} = \frac{dx}{x}$$

(Solve it by variable separable form and replace $v$ by $y/x$).